

Transitive orientations in bull-reducible Berge graphs

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Abstract

A bull is a graph with five vertices r, y, x, z, s and five edges ry, yx, yz, xz, zs . A graph G is bull-reducible if no vertex of G lies in two bulls. We prove that every bull-reducible Berge graph G that contains no antihole is weakly chordal, or has a homogeneous set, or is transitively orientable. This yields a fast polynomial time algorithm to color exactly the vertices of such a graph.

1 Introduction

A graph is *perfect* if for every induced subgraph H of G the chromatic number of H is equal to its clique number. Perfect graphs were defined by Claude Berge [1]. The study of perfect graphs led to several interesting and difficult problems. The first one is their characterization. Berge conjectured that a graph is perfect if and only if it contains no odd hole and no odd antihole, where a hole is a chordless cycle of length at least 5, and an antihole is the complementary graph of a hole. It has become customary to call *Berge graph* any graph that contains no odd hole and no antihole, and to call the above conjecture the “Strong Perfect Graph Conjecture”. This conjecture was proved by Chudnovsky, Robertson, Seymour, and Thomas [3]. A second problem is the existence of a polynomial-time algorithm to color optimally the vertices of a perfect graph, solved by Grötschel, Lovász and Schrijver [12] with an algorithm based on the ellipsoid method for linear programming. A third problem is the existence of a polynomial-time algorithm to decide if a graph is Berge, solved by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [2]. There remains a number of interesting open problems in the context of perfect graphs, among them the existence of a combinatorial algorithm to compute the chromatic number of a perfect graph.

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A *bull* is a graph with five vertices r, y, x, z, s and five edges ry, yx, yz, xz, zs ; see Figure 1. We will frequently use the notation $r-yxz-s$ for such a graph.

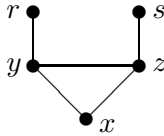


Figure 1: The bull $r-yxz-s$.

Bull-free Berge graphs have been much studied as a self-complementary class of Berge graphs for which first the “Strong Perfect Graph Conjecture” was established by Chvátal and Sbihi [4]; subsequently a polynomial-time recognition algorithm for bull-free Berge graphs was found by Reed and Sbihi [19], and further study of the structure of the class by De Figueiredo, Maffray and Porto [6, 7] and Hayward [14] led to a polynomial-time algorithm to color optimally the vertices of a bull-free Berge graph [8].

The goal of the present paper is to contribute to the search for a combinatorial algorithm to compute the chromatic number of a perfect graph by generalizing the results on the structure of bull-free Berge graphs [6] to the larger class of bull-reducible Berge graphs. A graph G is called *bull-reducible* if every vertex of G lies in at most one bull of G . Clearly, bull-free graphs are bull-reducible. Everett, de Figueiredo, Klein and Reed [5] proved that every bull-reducible Berge graph is perfect. Although this result now follows directly from the Strong Perfect Graph Theorem [3], the proof given in [5] is much simpler and leads moreover to a polynomial-time recognition algorithm for bull-reducible Berge graphs whose complexity is lower than that given for all Berge graphs in [2].

A graph is called *weakly chordal* (or “weakly triangulated”) if it contains no hole and no antihole. Hayward [13] proved that all weakly triangulated graphs are perfect, and there are very efficient algorithms to find an exact coloring for weakly chordal graphs [15, 17]. Given a subset of vertices S , a vertex is said to be *partial* on S if it has at least one neighbour and at least one non-neighbour in S . A vertex is *impartial* on S if it either sees all vertices of S or misses all vertices of S . A proper subset H of vertices is called *homogeneous* if it has at least two vertices and every vertex not in H sees either all or none of H , in other words every vertex not in H is impartial on H . Notice that if H is a homogeneous set of G then it is also a homogeneous set of the complement graph \overline{G} . A graph is called *transitively orientable* if it admits a *transitive orientation*, i.e., an orientation of its edges with no circuit and with no P_3 abc with the orientation \vec{ab} and \vec{bc} . Here we prove:

Theorem 1 *Let G be a bull-reducible Berge graph that contains no antihole. Then G either is weakly chordal, or has a homogeneous set, or is transitively orientable.*

Using Theorem 1, we can devise a polynomial-time algorithm that colors the vertices of any bull-reducible Berge graph that contains no antihole. This question will be addressed in Section 4.

A *wheel* is a graph made of an even hole of length at least 6 plus a vertex that sees all vertices of this hole. A *double broom* is a graph made of a P_4 plus two non-adjacent vertices a, a' that see all vertices of the P_4 , plus a vertex b that sees only a and a vertex b' that sees only a' . A *lock* is a graph with six vertices such that the first four induce a hole, the fifth one is adjacent to the first four, and the sixth one is adjacent to two adjacent vertices of the hole only. Given a graph F , a *spiked F* is a graph that consists in a copy of F plus two additional vertices a, b such that b has no neighbour in F and a is adjacent to every vertex of $V(F) \cup \{b\}$. Let us use the notation F_1, F_2 for the following two types of graphs: F_1 stands for the bull, and F_2 for the lock. Let \mathcal{B} be the class of bull-reducible Berge graphs that contain no wheel, no double broom, and no spiked F_j ($j = 1, 2$).

Theorem 2 *Let G be a graph in \mathcal{B} . If G contains a hole of length at least six and no antihole then G is transitively orientable.*

Lemma 1 ([5, 9]) *Let G be a bull-reducible C_5 -free graph. If G contains a wheel or a double broom then G has a homogeneous set.*

Lemma 2 *Let G be a bull-reducible C_5 -free graph. If G contains a spiked F_j for any $j = 1, 2$, then G has a homogeneous set.*

Lemma 2 is proved in Section 2, and Theorem 2 is proved in Section 3. We can see immediately how to obtain a proof our main Theorem.

Proof of Theorem 1. Let G be a bull-reducible Berge graph containing no antihole. If G contains a wheel, a double broom, or a spiked F_i ($i = 1, 2$) then G has a homogeneous set by Lemmas 1 and 2. So we may assume that G is a graph in the class \mathcal{B} . If G contains no hole of length at least six then it is weakly chordal. So we may assume that G contains a hole of length at least six. Then Theorem 2 implies that G is transitively orientable. \square

2 Some lemmas

Proof of Lemma 2. Suppose that G has an induced subgraph S that is a spiked F_j for some $j = 1, 2$. Let W be the set of vertices that induces the

F_j contained in S ; let b be the vertex of S that misses every vertex of W ; and let a be the vertex of S that sees all of $W \cup \{b\}$. Let W have vertices $u_1, \dots, u_{|W|}$ with the following notation. If W induces a bull (F_1), then it is $u_1-u_2u_5u_3-u_4$. If W induces a lock (F_2), it has edges $u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_5u_i$ ($i = 1, \dots, 4$), u_1u_6, u_2u_6 . In either case, we define additional sets of vertices as follows. Let T be the set of vertices of $G - W$ that see all vertices of W . Let Z be the set of vertices of $G - W$ that see none of W . Let P be the set of vertices of $G - W$ that have a neighbour and a non-neighbour in W . Clearly, W, T, Z, P form a partition of $V(G)$.

For any $p \in P$, say that three vertices $u, v, w \in W$ form a *blue triple* if they induce a subgraph with only one edge, say the edge uv , and p sees u and misses v, w . Note that if there is such a triple and p misses any $t \in T$ then $p-uvt-w$ is a bull in G , and we call any such bull a “blue bull”. Thus,

(1) *If p has two blue triples, then it sees all of T .*

For any $p \in P$, say that a chordless path $u-v-w$ of three vertices of W is *red* if p sees u, v and misses w . Note that if there is such a path and p sees any $z \in Z$ then $z-puv-w$ is a bull in G , and we call any such bull a “red bull”. Thus,

(2) *If p has two red paths, then it misses all of Z .*

We will need a classification of the vertices of P . Let p be a vertex in P ; then, using only the fact that $W \cup \{p\}$ induces a subgraph that contains no C_5 and at most one bull, it is a routine matter to establish that p must be of exactly one of the types listed in the following table:

W	$N(p) \cap W$ (up to symmetry)	number of blue triples	number of red paths
F_1	$\{u_1\}$ or $\{u_1, u_3\}$	1	0
	$\{u_1, u_2, u_5\}$ or $\{u_1, u_2, u_4, u_5\}$	0	1
	$\{u_2, u_5\}$	2	1
	$\{u_1, u_2, u_3\}$	1	2
F_2	$\{u_1\}$	1	0
	$\{u_3\}$ or $\{u_1, u_3\}$	2	0
	$\{u_3, u_5\}, \{u_1, u_3, u_5\}$ or $\{u_1, u_3, u_5, u_6\}$	≥ 1	1
	other possible types	–	≥ 2

The “other possible types” for F_2 are (up to symmetry):

$\{u_1, u_2\}, \{u_1, u_6\}, \{u_1, u_2, u_4\}, \{u_1, u_2, u_6\}, \{u_1, u_3, u_4\}, \{u_1, u_3, u_6\}, \{u_1, u_5, u_6\}, \{u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4\}, \{u_1, u_2, u_4, u_6\}, \{u_1, u_2, u_5, u_6\}, \{u_1, u_3, u_4, u_5\}, \{u_3, u_4, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4, u_6\}, \{u_1, u_2, u_4, u_5, u_6\}$, and $\{u_1, u_3, u_4, u_5, u_6\}$.

Let P_1 be the set of vertices p of P such that $N(p) \cap W$ is a stable set, and let $P_2 = P - P_1$. We claim that:

(3) *Every vertex of T sees every vertex of P_1 .*

For suppose on the contrary that there are non-adjacent vertices $t \in T$ and $p \in P_1$. By the remark, p has no red path. By (1), p has at most one blue triple. The table shows two types that satisfy these conditions, on the 1st line of F_1 and the 1st line of F_2 . In either case there is a blue bull. If W induces an F_1 then W and the blue bull intersect, a contradiction. If W induces an F_2 then $p-u_1u_6u_2-u_3$ is a second bull containing p , a contradiction. Therefore (3) holds.

Let A be the set of those vertices of T that have a neighbour in Z . Clearly $b \in Z$ and $a \in A$. We claim that:

(4) *Every vertex of A sees every vertex of P .*

For suppose on the contrary that some vertex $t \in A$ misses some vertex $p \in P$. Up to renaming vertices we may assume that $t = a$. Since $A \subseteq T$, and by (3), we have $p \in P_2$, so there is at least one red path for p . Suppose that p sees b . Then (2) implies that there is exactly one red path, and there is a red bull. Thus we must have zero blue triple for p . The table shows one type that satisfies these conditions, on the 2nd line of F_1 . In this case, W and the red bull are two intersecting bulls, a contradiction. Thus p misses b . Say that an edge uv with $u, v \in W$ is a *switch* if p sees u and misses v . Note that if there is such an edge then $b-avu-p$ is a bull in G (a “switch bull”). Thus, there must be at most one switch. In fact there is a switch since W is connected and both $W \cap N(p)$ and $W - N(p)$ are non empty; so there is a switch bull, and consequently there must be zero blue triple for p . If W induces an F_2 , then W is 2-connected and so there are two switches, a contradiction. If W induces an F_1 then the switch bull and W itself are two intersecting bulls, a contradiction. Therefore (4) holds.

Let X be the set of vertices x of Z such that there exists in G a path $x_0-x_1-\dots-x_k$ with $x_0 \in P$, $k \geq 1$, $x_1, \dots, x_k \in Z$, and $x = x_k$. We claim that:

(5) *Every vertex of A sees every vertex of X .*

For suppose that some vertex $t \in A$ misses some vertex $x \in X$. Up to renaming vertices we may assume that $t = a$. By the definition of X there is a path $x_0-\dots-x_k$ with $x_0 \in P$, $x_1, \dots, x_k \in Z$, and $x = x_k$. We may assume that k is minimal, so this path is chordless. By (4), a sees x_0 . Since W is connected in \overline{G} , there are non-adjacent vertices $w, w' \in W$ such that x_0 sees w and misses w' . Suppose that $k = 1$. Then x_1-x_0wa-w' is a bull in G , so there must be only one such pair w, w' . When W induces an F_1 , W itself is a second bull containing w , a contradiction. So let W induce F_2 . If $x_0 \in P_1$ then the table shows that $N(x_0) \cap W \subseteq \{u_1, u_3\}$, and either $x_0-u_1u_6u_2-u_3$ (if x_0 misses u_3) or $x_0-u_3u_5u_2-u_6$ (if x_0 sees u_3) is a second

bull containing x_0 , a contradiction. If $x_0 \in P_2$, then there is a red path in W , so there is a red bull with x_0, x_1 , which is a second bull containing x_0 , a contradiction. Now suppose that $k \geq 2$. The minimality of k implies that a sees x_{k-2} and x_{k-1} . Let w' be any vertex in $W - N(x_{k-2})$. Then $w'-ax_{k-2}x_{k-1}x_k$ is a bull containing x_0 for each choice of w' , which is a contradiction if $|W - N(x_{k-2})| \geq 2$. So we must have $|W - N(x_{k-2})| = 1$, which implies $k = 2$ and, by (3), $x_0 \in P_2$; but then there is a red path and a red bull with x_0, x_1 , which is a second bull containing x_0 , a contradiction. Therefore (5) holds.

Let Y be the set of vertices of $T - A$ such that there exists in \overline{G} a path $x-y_1 \cdots y_\ell$ with $x \in P \cup X$, $\ell \geq 1$, $y_1, \dots, y_\ell \in T - A$ and $y = y_\ell$. We claim that:

(6) *Every vertex of A sees every vertex of Y .*

For suppose that some vertex $t \in A$ misses a vertex $y \in Y$. Up to renaming vertices we may assume that $t = a$. By the definition of Y and X , there is a sequence of vertices $x_0, x_1, \dots, x_k, y_1, \dots, y_\ell$ such that $x_0x_1 \cdots x_k$ is a path in G and $x_ky_1 \cdots y_\ell$ is a path in \overline{G} such that $x_0 \in P$, if $k \geq 1$ then $x_1, \dots, x_k \in X$, $\ell \geq 1$, $y_1, \dots, y_\ell \in Y$ and $y = y_\ell$. We may assume that this sequence is minimal with these properties. Note that y_1, \dots, y_ℓ miss b since they are in Y . If $\ell \geq 2$, then, by the minimality of the sequence, a sees $y_{\ell-1}$, and so $b-ay_{\ell-1}w-y_\ell$ is a bull for each $w \in W$, a contradiction. So $\ell = 1$. If $k \geq 2$, then, by the minimality of the sequence, y_1 sees x_1 , which contradicts the definition of Y . So $k \leq 1$.

Suppose that $k = 0$. Then y_1 misses x_0 , which implies, by (3), that $x_0 \in P_2$ (so there is a red path) and, by (1), that x_0 has at most one blue triple. If x_0 misses b , then $b-ax_0w-y_1$ is a bull for each neighbour w of x_0 in W , which is possible only if there is only one such w , so $x_0 \in P_1$, a contradiction. Thus x_0 sees b . Then (2) implies that there is at most one, and so exactly one, red path for x_0 , and there is one red bull with x_0, b . Consequently there must be zero blue triple. The table shows one type that satisfies these conditions, on the 2nd line F_1 . In this case, W and the red bull are two intersecting bulls, a contradiction.

Suppose that $k = 1$. By the minimality, y_1 sees x_0 . Since x_0 has a neighbour x_1 in Z , (2) implies that there is at most one red path for x_0 . If there is one red path $u-v-w$, then there is a red bull x_1-x_0uv-w and a second bull $x_1-x_0uy_1-w$, a contradiction. So there is no red path. The table shows three types that satisfy this condition: on the 1st line of F_1 , and the 1st and 2nd lines of F_2 . In either case let w, w' be any two non-adjacent vertices of W such that x_0 sees w and misses w' . Such a pair exists since W is connected in \overline{G} . Then $x_1-x_0wy_1-w'$ is one bull. So there must be only one such pair w, w' . This implies that W must not be 2-connected in \overline{G} . If W is F_1 ,

then $N(x_0) \cap W$ is one of $\{u_1\}$ or $\{u_1, u_3\}$ (1st line) and there are also two pairs (u_1, u_4) and (u_1, u_5) like w, w' . So it must be that W induces an F_2 and $N(x_0) \cap W$ is one of $\{u_1\}$, $\{u_3\}$, or $\{u_1, u_3\}$. If it is $\{u_3\}$ then again there are two pairs (u_3, u_1) and (u_3, u_6) like w, w' ; if it is $\{u_1\}$, then $x_0-u_1u_6u_2-u_3$ is a second bull containing x_0 , and if it is $\{u_1, u_3\}$ then $x_0-u_3u_5u_2-u_6$ is a second bull, a contradiction. Therefore (6) holds.

Now $V(G)$ can be partitioned into the set $H = W \cup P \cup X \cup Y$ and the sets $T - Y$ and $Z - X$. We claim that:

(7) *Every vertex $h \in H$ sees every vertex $t \in T - Y$.*

If $h \in W$, this is by the definition of T . If $h \in P \cup X \cup Y$ and $t \in A$ this is by (4), (5) and (6). If $h \in P \cup X \cup Y$ and $t \in T - Y - A$ this is by the definition of Y . Thus (7) holds.

Next we claim that:

(8) *Every vertex $h \in H$ misses every vertex in $Z - X$.*

If $h \in W$ this is by the definition of Z . If $h \in P \cup X$ this is by the definition of X . If $h \in Y$ this is by the definition of Y ($\subseteq T - A$). Thus (8) holds.

Now it follows from (7), (8) and the fact that $T - Y$ is not empty (since it contains a) that H is a homogeneous set of G , which concludes the proof of the lemma. \square

We finish this section by recalling a useful lemma.

Lemma 3 ([5]) *Let G be a bull-reducible odd hole-free graph, and let C be a shortest even hole of length at least six in G , with its vertices labelled alternately “odd” and “even”. Let v be any vertex in $V(G) \setminus V(C)$. Then v satisfies exactly one of the following conditions:*

- $N(v) \cap V(C) = V(C)$, so C and v form a wheel;
- $N(v) \cap V(C)$ consists in either all even vertices and no odd vertex of C or all odd vertices and no even vertex of C ;
- $N(v) \cap V(C)$ consists in either one, or two consecutive or three consecutive vertices of C ;
- $N(v) \cap V(C)$ consists in two vertices at distance 2 along C ;
- C has length 6 and $N(v) \cap V(C)$ consists in four vertices such that exactly three of them are consecutive.

3 Transitive orientations

In a graph G , for any set $B \subseteq V(G)$, let $P(B)$ be the set of vertices of $V(G) - B$ that have a neighbour and a non-neighbour in B . Let us say that a graph G has a box partition if its vertex set can be partitioned into non-empty subsets, called boxes, with the following properties:

- (i) *Every box is labelled either odd or even, and there is no edge between two boxes that have the same label.*
- (ii) *Every box induces a connected subgraph of G .*
- (iii) *Boxes are either central or peripheral, and there are at least six central boxes.*
- (iv) *For each central box B there are four auxiliary vertices $a_B, b_B, c_B, d_B \in V(G) - B$, such that a_B and c_B see all of B and miss all of $P(B)$, b_B and d_B miss all of B , and the four auxiliary vertices induce a graph with two edges $a_B b_B, c_B d_B$.*
- (v) *For each peripheral box B there are two auxiliary vertices $a_B, b_B \in V(G) - B$, such that a_B sees all of B and misses all of $P(B)$, b_B misses all of B , and the two auxiliary vertices are adjacent.*
- (vi) *For any two adjacent vertices u, v in a peripheral box B the sets $N(u) \cap P(B)$ and $N(v) \cap P(B)$ are comparable by inclusion.*
- (vii) *A box B does not contain a chordless path $u-v-w$ such that there are adjacent vertices $x, y \in P(B)$ such that x sees u and misses v, w and y sees u, v and misses w .*

Lemma 4 *Let G be any bull-reducible, C_5 -free graph, and let $B \subseteq V(G)$ be a set that satisfies (iv) or (v). Suppose that there are vertices $u, v \in B$ and $x, y \in P(B)$ such that x sees u and misses v and y sees v and misses u . Then there is no P_4 induced by x, u, v, y . If u, v are adjacent, then G contains a \overline{C}_6 .*

Proof. Suppose x, u, v, y induce a P_4 . Since B satisfies (iv) or (v), then, with the same notation, we know that a_B sees u, v and misses x, y , and b_B misses u, v . $x-u a_B v-y$ is a bull. If b_B misses x then $b_B-a_B v u-x$ is a second bull, which intersects the first. So b_B sees x , and similarly it sees y . But then b_B, x, u, v, y induce a C_5 , a contradiction. Now suppose u, v are adjacent. This implies x, y are adjacent. Then b_B sees at least one of x, y , for otherwise $b_B-a_B u v-y$ and $b_B-a_B v u-x$ are two intersecting bulls. Then b_B sees both x, y , for if it sees only one, say x , then a_B, b_B, x, y, v induce a C_5 . Thus a_B, b_B, u, v, x, y induce a \overline{C}_6 . This completes the proof of the Lemma. \square

Let us say that a vertex x in a graph G is *sensitive* if there exist six vertices u_1, \dots, u_6 of $G - x$ with edges $u_i u_{i+1}$ ($i = 1, \dots, 5$) and possibly $u_1 u_6$ (so that they induce a P_6 or C_6) such that x is adjacent to u_2 and u_3 and not to u_1, u_4, u_5, u_6 , and $G - x$ contains a hole.

Theorem 3 *Let G be a graph in \mathcal{B} . If G contains a hole of length at least 6 and G has no sensitive vertex, then G admits a box partition.*

Proof. Let ℓ be the length of a shortest even hole of length at least 6 in G . So there exist ℓ non-empty disjoint subsets V_1, \dots, V_ℓ such that each vertex in V_i sees every vertex in $V_{i-1} \cup V_{i+1}$ and misses every vertex in $V_{i+2} \cup V_{i+3} \cup \dots \cup V_{i-2}$ (with subscripts modulo ℓ). Put $V_1^* = V_1 \cup V_3 \cup \dots \cup V_{\ell-1}$, $V_2^* = V_2 \cup V_4 \cup \dots \cup V_\ell$, and $V^* = V_1^* \cup V_2^*$. We may assume that V^* is maximal with this property. Let us then define the following subsets of vertices:

- Let A_1^* be the set of vertices that see all of V_2^* and miss all of V_1^* ;
- Let A_2^* be the set of vertices that see all of V_1^* and miss all of V_2^* ;
- For $i = 1, \dots, \ell$, let X_i be the set of all vertices not in $A_1^* \cup A_2^*$ that see all of $V_{i-1} \cup V_{i+1}$ and miss all of $V_{i-2} \cup V_{i+2}$;
- $D_i = V_i \cup X_i$;
- $D_1^* = D_1 \cup D_3 \cup \dots \cup D_{\ell-1}$, $D_2^* = D_2 \cup D_4 \cup \dots \cup D_\ell$;
- $C_1^* = D_1^* \cup A_1^*$, $C_2^* = D_2^* \cup A_2^*$;
- $Z = V(G) - (D_1^* \cup D_2^* \cup A_1^* \cup A_2^*)$.

Clearly, the sets $D_1, \dots, D_\ell, A_1^*, A_2^*, Z$ form a partition of $V(G)$. Note that subscripts on the starred sets are modulo 2, while subscripts on the unstarred sets are modulo ℓ . From now on we reserve the letter v_i for an arbitrary vertex in V_i ($i = 1, \dots, \ell$). Let us establish a number of useful facts.

- (1) *If any X_i is non-empty, then $\ell = 6$. Every vertex of X_i has a neighbour in V_{i+3} . If a vertex of X_i sees all of V_{i+3} then it has a neighbour in V_i .*

For simpler notation put $i = 3$. Let x be any vertex of X_3 . So x sees all of $V_2 \cup V_4$ and misses all of $V_1 \cup V_5$. Then x must have a neighbour in $V_6 \cup \dots \cup V_\ell$, for otherwise we could add x to V_3 , which would contradict the maximality of V^* . Let h be the smallest index such that x has a neighbour y in V_h with $6 \leq h \leq \ell$. If $h \geq 7$, then $\{x, v_4, \dots, v_{h-1}, y\}$ induces a hole of length $h - 2$, with $5 \leq h - 2 \leq \ell - 2$, which contradicts G being Berge (if h is

odd) or the definition of ℓ (if h is even). So $h = 6$. Suppose $\ell \geq 8$. Then we can apply Lemma 3 to the hole induced by $\{v_1, v_2, v_3, v_4, v_5, y, \dots, v_\ell\}$ and to x , which implies that x sees every v_j with even $j \neq 6$ and misses every v_j with odd j . Then applying Lemma 3 to the hole induced by $\{v_1, \dots, v_\ell\}$ implies that x also sees every $v_6 \in V_6$. But then we have $x \in A_1^*$, which contradicts the definition of X_3 . Thus $\ell = 6$. Now if x also sees all of V_6 and none of V_3 , then x must be in A_1 , which contradicts the definition of X_3 . So if x sees all of V_6 it has a neighbour in V_3 . Therefore (1) holds.

(2) *For i, j of the same parity, there is no edge between D_i and D_j .*

For if $\ell \geq 8$, this follows immediately from the fact that $D_i = V_i$ and $D_j = V_j$. Now let $\ell = 6$ and suppose up to symmetry that there is an edge xy with $x \in D_1$ and $y \in D_3$. Since x has a neighbour in D_3 we have $x \notin V_1$, so $x \in X_1$; and then, by (1), we have $\ell = 6$ and x has a neighbour $u_4 \in V_4$. Likewise, y is in X_3 and has a neighbour $u_6 \in V_6$. If x has a non-neighbour $w_4 \in V_4$ and y has a non-neighbour $w_6 \in V_6$ then $\{x, y, w_4, v_5, w_6\}$ induces a C_5 , a contradiction. So we may assume, up to symmetry, that x sees all of V_4 . Then (1) implies that x has a neighbour $w_1 \in V_1$. So we find a bull $w_1-xyu_4-v_5$. If y has a neighbour $w_3 \in V_3$, then we find a second bull $w_3-yxu_6-v_5$ containing x , a contradiction. So y has no neighbour in V_3 , and, by (1), y has a non-neighbour $w_6 \in V_6$. But then we find a second bull $v_5-w_6w_1x-y$, a contradiction. Therefore (2) holds.

(3) *There is no edge between A_1^* and D_1^* .
There is no edge between A_2^* and D_2^* .*

For suppose, up to symmetry, that a vertex a in A_1^* sees a vertex x_1 in D_1 . By the definition of A_1^* , x_1 is in X_1 , so $\ell = 6$ by (1), and x_1 has a neighbour $w_4 \in V_4$. If x_1 sees any $v_1 \in V_1$ then $v_1-x_1aw_4-v_3$ and $v_1-x_1aw_4-v_5$ are two intersecting bulls, a contradiction. If x_1 misses every $v_1 \in V_1$, then by (1) it misses some $u_4 \in V_4$. But then $v_1-v_2x_1a-u_4$ and $v_1-v_6x_1a-u_4$ are two intersecting bulls, a contradiction. Therefore (3) holds.

(4) *Each $z \in Z$ misses all of V_1^* or all of V_2^* , and there is at most one $i \in \{1, \dots, l\}$ such that z sees all of V_i .*

To prove the first part of the claim, suppose on the contrary and up to symmetry that z has neighbours $w_1 \in V_1$ and $w_j \in V_j$ for some even j . First suppose that $j \in \{2, \ell\}$, say (up to symmetry) $j = 2$. Pick any $w_h \in V_h$ for $h = 3, \dots, \ell$. Then w_1, \dots, w_ℓ induce a hole. If w_1, w_2 are the only neighbors of z in that hole, then z is a sensitive vertex, a contradiction. So z has at least three vertices in that hole and, by Lemma 3, z must see exactly one of w_ℓ, w_3 , say z sees w_ℓ , and then miss all of $w_3, \dots, w_{\ell-1}$ (if $\ell \geq 8$) or miss w_3, w_5 (if $\ell = 6$). Repeating this argument for every choice of w_h with $h \neq 1$, we obtain that z sees all of $V_\ell \cup V_2$ and misses all of $V_{\ell-1} \cup V_3$. Since z has

a neighbour in V_1 , z must be in D_1 , a contradiction. Now suppose that $4 \leq j \leq l-2$. Pick any $w_h \in V_h$ for $h = 2, \dots, \ell$, $h \neq j$. Then w_1, \dots, w_ℓ induce a hole. By Lemma 3 and up to symmetry, we must have $\ell = 6$, $j = 4$, and z must see both w_6, w_2 and miss both w_5, w_3 . Then repeating this argument for every choice of w_h with $h \in \{2, 3, 5, 6\}$ implies that z sees all of $V_6 \cup V_2$ and misses all of $V_5 \cup V_3$. Since z has a neighbour in V_4 , z must be in D_1 , a contradiction. Thus we have prove the first part of the claim. To prove the second part, suppose on the contrary that z sees all of $V_i \cup V_j$ for some $i \neq j$. So i, j have the same parity. If $j = i+2$, then z should be in $D_{i+1} \cup A_1^* \cup A_2^*$, a contradiction. Likewise for $j = i-2$. So $i+4 \leq j \leq i-4$, where indices are taken modulo ℓ , which contradicts Lemma 3. Therefore (4) holds.

(5) *Each vertex of Z misses all of C_1^* or all of C_2^* .*

For suppose that z has neighbours $x \in C_1^*$ and $y \in C_2^*$. Up to symmetry there are two cases: (a) $x \in D_1 \cup A_1^*$ and $y \in D_2 \cup A_2^*$; and (b) $x \in D_1$ and $y \in D_j$ with $4 \leq j \leq l-2$. In either case, by (4) we can pick vertices $w_h \in V_h$ for $h = 1, \dots, \ell$ such that z sees at most one of them.

Consider case (a). Suppose that x, y are adjacent. If z sees w_4 , then $w_\ell - xzy - w_3$ is a bull; if x misses w_4 , then $w_\ell - xyz - w_4$ is a second bull, while if x sees w_4 , then $w_\ell - xzw_4 - w_3$ is a second bull, a contradiction. So z misses w_4 . Likewise z misses $w_{\ell-1}$. Suppose that z sees w_3 . Then $w_1 - yzw_3 - w_4$ is a bull. Then x misses w_1 , for otherwise $w_{\ell-1} - w_\ell w_1 x - z$ is a second bull. Then x misses w_4 , for otherwise $w_1 - yzx - w_4$ is a second bull. Then y misses w_2 , for otherwise $w_\ell - w_1 w_2 y - z$ is a second bull. Then y misses w_5 , for otherwise $w_5 - yzx - w_2$ is a second bull. Then y misses w_4 , for otherwise $w_2 - xzy - w_4$ is a second bull. But now vertices $w_1, y, w_3, w_4, \dots, w_\ell$ induce a hole, and the neighbors of z in that hole are y and w_3 , so z is a sensitive vertex, a contradiction. So z misses w_3 . Similarly z misses w_ℓ . Thus $w_\ell - xzy - w_3$ is a bull. If x has a non-neighbour $u_4 \in V_4$ and y has a non-neighbour $u_{\ell-1} \in V_{\ell-1}$ then $x, y, w_3, u_4, \dots, u_{\ell-1}, w_\ell$ induce a hole, and the neighbors of z in that hole are x and y , so z is a sensitive vertex, a contradiction. So we may assume up to symmetry that x sees every vertex of V_4 , which, by (1), implies that $\ell = 6$ and x has a neighbour $u_1 \in V_1$. But then if z sees u_1 , then $w_6 - u_1 zy - w_3$ is a second bull containing z , while if z misses u_1 , then $w_5 - w_6 u_1 x - z$ is a second bull containing z , a contradiction. Thus x, y are not adjacent. This implies that $x \in X_1$ and $y \in X_2$, so $\ell = 6$ and x has a neighbour $u_4 \in V_4$ and y has a neighbour $u_5 \in V_5$. Then z sees one of u_4, u_5 , for otherwise z, x, u_4, u_5, y induce a C_5 . Up to symmetry z sees u_4 . Then (4) implies that z misses all of w_1, w_3, w_5 . If z misses $w_6 \in V_6$, then $w_6 - xzu_4 - w_3$ and $w_6 - xu_4 z - y$ are two intersecting bulls, a contradiction. So z sees w_6 , and it misses w_2 . But then $w_5 - w_6 zx - w_2$ and $w_5 - u_4 zx - w_2$ are two intersecting bulls, a contradiction.

Now consider case (b). By (4), we have either $x \in X_1$ or $y \in X_j$, and so

$\ell = 6$ and $j = 4$. Then, up to symmetry, z misses w_3, w_5, w_6 . Then x, y are adjacent, for otherwise z, y, w_5, w_6, x induce a C_5 . Then $w_6-xzy-w_3$ is a bull. If z misses w_2 , then $w_2-xzy-w_5$ is a second bull, while if z sees w_2 , then $w_6-xzw_2-w_3$ is a second bull, in either case a contradiction. Therefore (5) holds.

We let Z_1^* (resp. Z_2^*) denote the set of vertices of Z that have a neighbour in C_2^* (resp. C_1^*). By Claim 5, $Z_1^* \cap Z_2^* = \emptyset$, there is no edge between Z_1^* and C_1^* , and there is no edge between Z_2^* and C_2^* .

Now, we decompose the whole graph into connected subsets based on a “hanging” from C_1^* . Precisely, let us define sets:

$$L_1 = C_1^*, \quad L_2 = N(L_1) = C_2^* \cup Z_2^*, \quad L_{j+1} = N(L_j) - L_{j-1}$$

for any $j \geq 2$, as long as this defines non-empty sets. The L_j ’s will be called the levels of the decomposition. Note that $Z_1^* \subseteq L_3$ by (5). Level L_i will be called odd or even according to the parity of i .

A vertex will be called *central* if it is in $C_1^* \cup C_2^*$, and *peripheral* otherwise. We will call *box* any subset that induces a connected component in any L_j . It is clear that the whole vertex set of the graph is partitioned into boxes. By Fact (5), a box may contain either only central vertices or only peripheral vertices. The boxes will be called central or peripheral accordingly. More precisely, by (2) and (3), every central box is a subset of some D_i or of some A_i^* . Level L_1 consists of central boxes only. Level L_2 consists of the central boxes in C_2^* , plus the peripheral boxes in Z_2^* (if any). The deeper levels consist of peripheral boxes exclusively. Clearly, Properties (i), (ii), (iii) hold. We now prove that each box in L_j satisfies the desired Properties (iv), (v), (vi) or (vii) by induction on j .

(6) *Every central box satisfies Properties (iv) and (vii).*

For let B be any central box. We may assume up to symmetry that $B \subseteq D_3$ or $B \subseteq A_1^*$. In either case every vertex of B sees all of $V_2 \cup V_4$ and misses all of $V_1 \cup V_5$. We claim that every $z \in P(B)$ misses all of $V_2 \cup V_4$. For suppose on the contrary, and up to symmetry, that z sees some $w_2 \in V_2$. There are adjacent vertices $u, v \in B$ such that z sees u and misses v . Suppose that z sees any $w_1 \in V_1$. Then (4) implies $z \in D_1 \cup D_2$. If $z \in D_1$, the edge zu contradicts (2) or (3). So $z \in D_2$, and so z misses all of $V_\ell \cup V_4$. Then $v_\ell-w_1zw_2-v$ is a bull. If z misses any $w_5 \in V_5$, then $z-uvv_4-w_5$ is a second bull containing z , while if z sees any $w_5 \in V_5$, $w_5-zw_1w_2-v$ is a second bull containing z , a contradiction. Thus z misses all of V_1 . Suppose that z also sees some $w_4 \in V_4$. Then by symmetry z misses all of V_5 . Vertex z cannot see all of $V_2 \cup V_4$, for otherwise z would be in $D_3 \cup A_1^*$, contradicting the

fact that $z \in P(B)$. So, up to symmetry, we may assume that z has a non-neighbour $w'_2 \in V_2$. But then $v_1-w'_2vu-z$ and $w'_2-uzw_4-v_5$ are two intersecting bulls, a contradiction. Thus z misses all of V_4 . Then $v_1-w_2zu-v_4$ is a bull. If z misses any $w_5 \in V_5$, then $z-uvv_4-w_5$ is a second bull containing z , while if z sees w_5 , then z, w_5, v_4, v, w_2 induce a C_5 , a contradiction. So we have proved that every vertex in $V_2 \cup V_4$ misses all of $P(B)$. Thus it suffices to take auxiliary vertices $a_B = v_2, b_B = v_1, c_B = v_4, d_B = v_5$ for B . Thus Property (iv) is established. To prove (vii), suppose on the contrary that there are vertices u, v, w, x, y as in the statement of (vii). Then v_1 sees one of x, y , for otherwise v_1-v_2vu-x and v_1-v_2wv-y are two intersecting bulls. Then v_1 sees y , for otherwise it sees x and then v_1, x, y, v, v_2 induce a C_5 . Likewise v_5 sees y . But then $v_1-yuv-w$ and $v_5-yuv-w$ are two intersecting bulls, a contradiction. So (vii) is established. Therefore (6) holds.

The preceding claims imply that all boxes in L_1 and all boxes in $L_2 - Z_2^*$ satisfy Properties (iv) and (vii). Now we consider the peripheral boxes, which are the boxes in Z_2^* and in L_j for any $j \geq 3$. First we consider the boxes in Z_2^* .

- (7) *Given non-adjacent vertices a, b , both in C_1^* , or both in $C_2^* \cup Z_2^*$, or both in Z_1^* , there exists a chordless even path R_{ab} whose interior vertices are alternately in L_1 and L_2 .*

For suppose first that a, b are both in $L_1 = C_1^*$. By the definition of the D_i 's and A_j^* 's, every vertex in any such set is adjacent to some vertex of V^* . Thus there is a path from a to b whose interior vertices are alternately in even V_i 's and odd V_i 's and no two interior vertices are in the same V_i . Take a shortest such path $R = a-v_h \cdots v_k-b$. Clearly, R has even length. Then (2) and (3) imply that any chord of R must be of the type av_i for some even $i > h$ or bv_j for some even $j < k$, and so $a-v_i \cdots v_k-b$ or $a-v_h \cdots v_j-b$ is a path with the same properties and shorter than R , a contradiction. Now suppose that a, b are both in $L_2 = C_2^* \cup Z_2^*$. Let a' be a neighbour of a in L_1 and b' be a neighbour of b in L_1 . If a, b have a common such neighbour, then we can take $a' = b'$ and $R_{ab} = a-a'-b$. In the remaining case, we may assume that a misses b' and b misses a' . If a', b' are adjacent, then they lie in one box in L_1 , for which Property (iv) is already proved, and then a, a', b, b' violate Lemma 4. So a', b' are not adjacent vertices in $L_1 = C_1^*$ and there exists a path $R_{a'b'}$ with the desired properties. Then the path $a-R_{a'b'}-b$ has even length and its interior vertices are in $L_1 \cup L_2$ and alternately in odd and even L_i 's. If this path has any chord, then it must be incident with a or b , and then (5) implies that we can find a shorter subpath with the same properties. Finally, suppose that a, b are both in Z_1^* . Note that the definition of the levels implies that the set Z_1^* is contained in L_3 . So, by considering a neighbour a' of a in L_2 and a neighbour b' of b in L_2 , and by applying an analogous argument we obtain a path with the desired

properties. Therefore (7) holds.

(8) *Every box in Z_2^* and every box in Z_1^* satisfy Properties (v), (vi), (vii).*

For let B be any box in Z_2^* . First let us prove the assertion that, for every subset $C \subseteq B$ that induces a connected subgraph, there is a vertex of L_1 that sees all vertices of C . We prove the assertion by induction on $|C|$. If $|C| = 1$ the assertion holds by the definition of B . Now suppose that the assertion holds for any C of size at most k , and let C have size $k + 1 \geq 2$. Let $c_1 \dots c_h$ be a longest chordless path in C . Thus $C - c_1$ and $C - c_h$ are connected and, by the induction hypothesis, there is a vertex $u \in L_1$ that sees all of $C - c_1$, and there is a vertex $v \in L_1$ that sees all of $C - c_h$. If u sees c_1 , or v sees c_h , then we are done. So let us assume that u misses c_1 and v misses c_h . Note that for each a in L_1 there is a vertex a' that sees a and misses all of B ; indeed, a is in $D_i \cup A_1^*$ for some odd i , and so any vertex in V_{i+1} can play the role of a' . In particular we can consider vertices u' and v' . If $h \geq 6$, then $c_1 - c_2 c_3 u - c_5$ and $c_1 - c_2 c_3 u - c_6$ are two intersecting bulls, a contradiction. If $3 \leq h \leq 5$, then $c_1 - c_2 c_3 u - u'$ and $c_h - c_{h-1} c_{h-2} v - v'$ are two intersecting bulls, a contradiction. So $h = 2$. This means that C is a clique. Suppose that u misses v . Consider any path $R_{uv} = r_1 \dots r_p$ given by (7), with p odd, $r_1 = u$, $r_p = v$. Then $v - c_1 - c_2 - u - R_{uv} - v$ is an odd cycle of length at least five, so it must contain a triangle, for otherwise it contains an odd hole. Note that c_1 and c_2 do not see two consecutive vertices on the path R_{uv} , since they are in Z_2^* and by (5). So, in order to have a triangle, there must be a vertex r_j that sees both c_1, c_2 , and so $r_j \in L_1$, and so $3 \leq j \leq p - 2$. But then $u - c_2 c_1 r_j - r_{j+1}$ and $v - c_1 c_2 r_j - r_{j-1}$ are two intersecting bulls, a contradiction. Thus u, v are adjacent, and so they lie in one box U of L_1 , and $c_1, c_2 \in P(U)$. Up to symmetry, we may assume that $U \subseteq D_3 \cup A_1^*$ and so, as proved in (6), v_2 sees all of U and misses all of $P(U)$ and v_1 misses all of U . Then v_1 sees one of c_1, c_2 , for otherwise $v_1 - v_2 uv - c_1$ and $v_1 - v_2 vu - c_2$ are two intersecting bulls. Then v_1 sees both c_1, c_2 , for if it sees only one, say c_1 , then v_1, v_2, u, c_2, c_1 induce a C_5 . Then v_1 sees every $z \in C - \{c_1, c_2\}$, for otherwise v_1, v_2, u, z, c_1 induce a C_5 (recall that C is a clique and u sees all of $C - c_1$). Thus we have proved the assertion. Applying it to $C = B$, we obtain that some vertex a of L_1 sees all of B .

Up to symmetry we may assume that $a \in D_3 \cup A_1^*$. Now we claim that a misses every vertex of $P(B)$. For suppose on the contrary that a sees some $x \in P(B)$. There are adjacent vertices $u, v \in B$ such that x sees u and misses v . Since x sees a , we have $x \in D_2 \cup D_4 \cup D_6 \cup A_2^* \cup Z_2^*$. In fact we do not have $x \in Z_2^*$, for otherwise x should be in B . Also we do not have $x \in D_6$, for otherwise u would contradict (5). Thus, up to symmetry, we have $x \in D_2 \cup A_2^*$. Since $u, v \in Z_2^*$, they both miss all of $V_2 \cup V_4$. Up to symmetry we may assume that u misses some $w_1 \in V_1$. Then $w_1 - xua - v_4$ is a bull. Then v misses w_1 , for otherwise $w_1 - vua - v_4$ is a second bull containing a . Suppose that x sees every $v_5 \in V_5$. If x sees any $w_2 \in V_2$, then v sees v_5 ,

for otherwise v_5-xv_2a-v is a second bull containing a , and then u sees every v_5 , for otherwise $v_5-vua-v_2$ is a second bull containing a . On the other hand if x misses any $w_2 \in V_2$, then u sees v_5 , for otherwise $v_2-aux-v_5$ is a second bull. In either case u sees all of V_5 . Since $u \in Z_2^*$, u must miss some $w_3 \in V_3$. But then $v_6-v_5ux-w_3$ is a second bull containing x , a contradiction. Thus x has a non-neighbour $w_5 \in V_5$. Suppose a has a non-neighbour $w_6 \in V_6$. If $\ell \geq 8$, let us pick any $w_i \in V_i$ for $i = 7, \dots, \ell$; then $w_1, v_2, a, v_4, w_5, \dots, w_\ell$ induce a hole in $G-u$, and $w_1-x-a-v_4-w_5-w_6$ is a P_6 or C_6 in $G-u$ such that u is adjacent to x, a and not to w_1, v_4, w_5, w_6 , so u is a sensitive vertex, a contradiction. So a sees all of V_6 , and so $a \notin V_3$. Consider any $v_3 \in V_3$. The same argument as for a implies that v_3 misses one of u, v . Then v_3 misses a , for otherwise $w_1-v_2v_3a-u$ or $w_1-v_2v_3a-v$ is a second bull. Then v_3 sees u , for otherwise $v_3-xua-v_6$ is a second bull. So v_3 misses v . But then $v_3-uva-v_6$ is a second bull containing a , a contradiction. Therefore a misses all of $P(B)$. So a can play the role of a_B , and any vertex in V_2 can play the role of b_B . Thus Property (v) is established.

In order to prove Property (vi), suppose on the contrary that there are two adjacent vertices $u, v \in B$ and two vertices $x, y \in P(B)$ such that x sees u and misses v and y sees v and misses u . Since B satisfies (v), by Lemma 4, x sees y . If x misses any $w_2 \in V_2$, then y also misses w_2 , for otherwise w_2, a, u, x, y induce a C_5 ; but then $w_2-avu-x$ and $w_2-auv-y$ are two intersecting bulls, a contradiction. Thus x sees all of V_2 . Similarly x sees all of V_4 . But then x should be in $D_3 \cup A_1^*$, a contradiction. Thus Property (vi) is established.

In order to prove (vii), suppose on the contrary that there are vertices u, v, w, x, y as in the statement of (vii). Then v_2 sees one of x, y , for otherwise $v_2-avu-x$ and $v_2-auv-y$ are two intersecting bulls. Then v_2 sees y , for otherwise it sees only x , and then v_2, x, y, v, a induce a C_5 . Likewise v_4 sees y . But then $v_2-yuv-w$ and $v_4-yuv-w$ are two intersecting bulls. So (vii) is established.

An analogous argument establishes the properties for a box in Z_1^* . Therefore (8) holds.

Now we consider the boxes in L_j for $j \geq 3$ that are not in Z_1^* .

- (9) *For any $j \geq 1$, given non adjacent vertices a, b in L_j , there exists a chordless even path R_{ab} whose interior vertices are in $L_1 \cup \dots \cup L_{\max\{2, j-1\}}$ and alternately in odd and even L_i 's. Moreover, every box in L_j with $j \geq 3$ satisfies Properties (v), (vi) and (vii).*

We prove this claim by induction on j . For $j = 1$, or $j = 2$, or $j = 3$ and a, b both in Z_1^* , this is Claims (6), (7), (8). Now suppose $j \geq 3$ and a, b not both in Z_1^* . Let a' be a neighbour of a in L_{j-1} and b' be a neighbour of b in L_{j-1} . If a, b have a common such neighbour, then we can take $a' = b'$ and $R_{ab} = a-a'-b$. In the remaining case, we may assume that a misses b' and b

misses a' . If a', b' are adjacent, then they lie in one box in L_{j-1} , for which Property (iv) or (v) is already proved, and then a, a', b, b' violate Lemma 4. So a', b' are not adjacent. By the induction hypothesis, there exists a path $R_{a'b'}$ with the desired properties. Then the path $a-R_{a'b'}-b$ has even length and its interior vertices are in $L_1 \cup \dots \cup L_{j-1}$ and alternately in odd and even L_i 's. If this path has any chord, then it must be incident with a or b , and then (5) implies that we can find a shorter subpath with the same properties.

Now the proof of Properties (v), (vi) and (vii) is rather similar to the proof of (8). First we prove that some vertex of L_{j-1} sees all of B , with the following changes: instead of (7), use the chordless even path R_{ab} given by induction; for every vertex a in L_{j-1} , there is a neighbour a' of a in L_{j-2} that misses all of B ; when (6) is invoked to deal with the C_4 induced by u, v, c_1, c_2 with $u, v \in U$, we can still invoke (6) if U is a central box, and we can invoke Property (vi) when U is a peripheral box, since that property holds for U by the induction hypothesis on j . Thus there is a vertex $a \in L_{j-1}$ that sees all of B . If a is in a central box, the rest of the proof is completely the same, with subscripts shifted by 1. There remains to deal with the case when a is in Z .

We prove that a misses all of $P(B)$. Suppose that a sees a vertex $x \in P(B)$. So there are adjacent vertices $u, v \in B$ such that x sees u and misses v . Since x is not in B , it is not in L_j ; and since it sees a and u , it must be in L_{j-1} . So a, x are in a box $U \subseteq L_{j-1}$, and this box has auxiliary vertices a_U, b_U by the induction hypothesis on j , with $a_U \in L_{j-2}$. Suppose that a_U is a central vertex, say $a_U \subseteq D_3 \cup A_1^*$ with $a \in Z_2^*$ (the case $a_U \in D_2 \cup A_2^*$ is similar). So $U \subseteq Z_2^*$, and so a, x miss all of V_2^* . Vertices u, v miss all of V_1^* since they are in L_3 . Since $u \in Z$, by (4) it has a non-neighbour in $V_2 \cup V_4$, say u misses $w_4 \in V_4$. Then v misses w_4 , for otherwise w_4, v, u, x, a_U induce a C_5 . So $v-axa_U-w_4$ is one bull. Then v sees every $v_2 \in V_2$, for otherwise $v-axa_U-v_2$ is a second bull; and u sees every v_2 , for otherwise v_2, v, u, x, a_U induce a C_5 . By (4), x misses some $w \in V_1 \cup V_3$. But then $w-v_2vu-x$ is a second bull. So a_U is not a central vertex, and so $j \geq 4$. By the definition of the levels, there is a shortest path $p_1 \dots p_r$ such that $p_r = a_U$ and p_1 is in $Z_1^* \cup Z_2^*$ (and so every vertex of $P \setminus p_1$ has no neighbour in $C_1^* \cup C_2^*$). By (4), there are vertices $w_i \in V_i$, $i = 1, \dots, \ell$, such that p_1 sees exactly one of them. If $j \geq 5$, then the subgraph of $G - x$ induced by vertices $w_1, \dots, w_\ell, p_1, \dots, p_r = a_U, a, v$ contains a hole and a P_6 $v-a-a_U \dots$ such that x sees a_U, a and misses the other four vertices of the P_6 , so x is a sensitive vertex, a contradiction. So $j = 4$, and so $a_U = p_1$. Since a, x are in L_3 they miss all of $w_1, w_3, \dots, w_{\ell-1}$. Suppose that some w_j with even j sees x ; then it sees a , for otherwise w_j-xa_U-a-v is a second bull; but then $w_{j-1}-w_jxa-v$ is a second bull, a contradiction. So x misses every w_i . But then the subgraph of $G - x$ induced by vertices $w_1, \dots, w_\ell, a_U, a, v$ contains

a hole and a P_6 $v-a-a_U-w_1-w_2-w_3$ such that x sees a_U, a and misses the other four vertices of the P_6 , so x is a sensitive vertex, a contradiction. Thus we have proved that a misses every vertex of $P(B)$. Since $j \geq 3$, a has a neighbour a' in L_{j-2} , and so a and a' can play the role of a_B and b_B , and Property (v) is established.

There remains to prove (vi) and (vii). In order to prove Property (vi), suppose on the contrary that there are two adjacent vertices $u, v \in B$ and two vertices $x, y \in P(B)$ such that x sees u and misses v and y sees v and misses u . Since B satisfies (v), by Lemma 4, x sees y . If a' misses both x, y , then $a'-auv-y$ and $a'-avu-x$ are two intersecting bulls; and if a' sees only one of x, y , then a', a, x, y and one of u, v induce a C_5 . So a' sees both x, y . Thus x, y are in one box B' , which is in L_{j-1} . If $j \geq 4$, then a' has a neighbour a'' in L_{j-3} , and $a''-a'xy-v$ and $a''-a'yx-u$ are two intersecting bulls. So $j = 3$, and x, y are in L_2 . If the box B' that contains x, y is peripheral, then the situation contradicts the fact that B' satisfies (vi), which was proved in (8). So B' is a central box. Up to symmetry we may assume that $B' \subset V_4 \cup X_4 \cup A_2^*$, and so, as in the proof of (6), we know that v_3 and v_5 see all of B' and every vertex in $V_2 \cup V_6$ misses all of B' . In consequence every w in V_2 sees both u, v (for if it misses both then $w-v_3xy-u$ and $w-v_3yx-u$ are two intersecting bulls, and if it sees only one then w, v_3, u, v and one of x, y induce a C_5); and similarly every w in V_6 sees both u, v . But then the fact that u, v see all of $V_2 \cup V_6$ contradicts the definition of the X_i 's, A_j^* 's and Z . Thus Property (vi) is established.

In order to prove (vii), suppose on the contrary that there are vertices u, v, w, x, y as in the statement of (vii). If a' misses both x, y , then $a'-avu-x$ and $a'-awv-y$ are two bulls. If a' sees x and not y , then a', x, y, v, a induce a C_5 . So a' sees y , and $a'-yuv-w$ is one bull. Then a' sees x , for otherwise $a'-avu-x$ is a second bull. So x, y are in one box B' in L_{j-1} . If $j \geq 4$, then a' has a neighbour a'' in L_{j-3} , and $a''-a'xy-v$ is a second bull, a contradiction. So $j = 3$. Since $a'-yuv-w$ is a bull for each neighbour a' of a in L_1 , this a' must be unique, so a is a peripheral vertex. Since a' is in L_1 , we may assume up to symmetry that it is in $V_3 \cup X_3 \cup A_1^*$. Then we may assume that x, y are different from and not adjacent to v_2 (else replace v_2 by v_4). Then v_2 sees v , for otherwise $v_2-a'xy-v$ is a second bull; v_2 sees u , for otherwise v_2, a', x, u, v induce a C_5 ; and v_2 sees w , for otherwise $a'-v_2uv-w$ is a second bull. Then v_1 sees x , for otherwise v_1-v_2vu-x is a second bull; and v_1 sees y , for otherwise v_1-v_2wv-y is a second bull. But then $v_1-yuv-w$ is a second bull. So (vii) is established. Therefore (9) holds.

This completes the proof of Theorem 3. \square

Lemma 5 *Let G be a bull-reducible graph that contains no C_5 , no wheel and no spiked bull. Suppose that G has a sensitive vertex x , and that $G - x$ is transitively orientable. Then G is transitively orientable.*

Proof. Since x is a sensitive vertex, there exist vertices u_1, \dots, u_6 of $G - x$ with edges $u_i u_{i+1}$ ($i = 1, \dots, 5$) and possibly $u_1 u_6$, such that x is adjacent to u_2 and u_3 and not to u_1, u_4, u_5, u_6 . Note that $u_1 - u_2 x u_3 - u_4$ is one bull, henceforth the “first bull”. (Every second bull we will find will obviously intersect the first one.) Define sets:

$$\begin{aligned} A &= \{v \in V(G) \mid v \text{ sees } x, u_2 \text{ and misses } u_3, u_5\}, \\ B &= \{v \in V(G) \mid v \text{ sees } x, u_3 \text{ and misses } u_1, u_2, u_4, u_6\}. \end{aligned}$$

We first claim that $N(x) = \{u_2, u_3\} \cup A \cup B$. To prove this, consider any neighbour v of x different from u_2, u_3 . Suppose that v misses both u_2, u_3 . Then v sees u_1 , for otherwise $u_1 - u_2 u_3 x - v$ is a second bull; and similarly v sees u_4 ; but then v, u_1, \dots, u_4 induce a C_5 . So v sees at least one of u_2, u_3 . Suppose that v sees both u_2, u_3 . If v sees u_4 , then it sees u_1 , for otherwise $u_1 - u_2 x v - u_4$ is a second bull; v sees u_5 , for otherwise $u_1 - v u_3 u_4 - u_5$ is a second bull; and v sees u_6 , for otherwise $u_2 - v u_4 u_5 - u_6$ is a second bull; but then, if $u_1 u_6$ is not an edge, then $u_1, \dots, u_4, x, v, u_6$ induce a spiked bull, and if $u_1 u_6$ is an edge, then v, u_1, \dots, u_6 induce a wheel. So v misses u_4 . Then v misses u_1 , for otherwise $u_1 - v x u_3 - u_4$ is a second bull. But then $u_1 - u_2 v u_3 - u_4$ is a second bull. Thus v sees exactly one of u_2, u_3 .

Now suppose that v sees u_2 and misses u_3 . Then v misses u_5 , for otherwise either $u_1 - u_2 x v - u_5$ (if v misses u_1) or $u_3 - u_2 u_1 v - u_5$ (if v sees u_1) is a second bull. Thus v is in A .

Now, suppose that v sees u_3 and misses u_2 . If v sees u_4 , then v sees u_5 , for otherwise $u_2 - u_3 v u_4 - u_5$ is a second bull; and v sees u_6 , for otherwise $x - v u_4 u_5 - u_6$ is a second bull; but then $u_2 - u_3 u_4 v - u_6$ is a second bull. So v misses u_4 . Then v misses u_1 , for otherwise $u_1 - v x u_3 - u_4$ is a second bull; and similarly v misses u_6 . Thus v is in B . So we have proved the claim that $N(x) = \{u_2, u_3\} \cup A \cup B$.

Next, we claim that every vertex in A sees every vertex in B . For suppose on the contrary that there are non-adjacent vertices $a \in A, b \in B$. Then a sees u_4 , for otherwise $a - x b u_3 - u_4$ is a second bull; but then $b - x u_2 a - u_4$ is a second bull, a contradiction. In summary, the two sets $A \cup \{u_3\}$ and $B \cup \{u_2\}$ form a partition of $N(x)$ and are completely adjacent to each other.

Let U_2 be the set of vertices that see u_1 and u_3 and miss u_4, u_5, u_6 . Note that x has only one neighbour (which is u_2) in U_2 , because for any such vertex w there is a bull $u_1 - w x u_3 - u_4$. Let D be the component of U_2 that contains u_2 . Let $N_2 = U_2 \cap N(u_2)$ and $M_2 = U_2 - (N_2 \cup \{u_2\})$. Then:

$$\begin{aligned} &\text{Every vertex of } N_2 \text{ sees every vertex of } M_2, \text{ and (consequently)} \\ &\text{either } D = \{u_2\} \text{ or } D = U_2. \end{aligned} \tag{1}$$

For consider any $v \in N_2$ and $w \in M_2$. Then v sees w , for otherwise $x - u_2 v u_1 - w$ is a bull. Therefore (1) holds.

If $P(D) - x \neq \emptyset$, then $M_2 = \emptyset$ and every vertex z of $P(D) - x$ satisfies one of the following:

- (a) z sees all of $\{x, u_1, u_3, u_5\} \cup N_2$ and none of $\{u_2, u_4, u_6\}$;
- (b) z sees all of $\{u_2, u_4\}$ and none of $\{x, u_1, u_3\} \cup N_2$.

(2)

To prove this, suppose that $P(D) - x \neq \emptyset$ and let z be any vertex in $P(D) - x$. So there are vertices u, v in D such that z sees u and misses v . By (1), we have $D = U_2$. So z is not in U_2 . First suppose that z sees both u_1, u_3 . If z sees u_4 , then it sees u_5 (for otherwise $u_1 - zu_3u_4 - u_5$ is a second bull), and it sees u_6 (for otherwise $u - zu_4u_5 - u_6$ is a second bull); but then $v - u_3u_4z - u_6$ is a second bull. So z misses u_4 . Then z misses u_6 , for otherwise $u_4 - u_3uz - u_6$ is a second bull. Then z sees u_5 , for otherwise z should be in U_2 . If x misses z , then x sees u , for otherwise $x - u_3uz - u_5$ is a second bull; but then $x - uu_1z - u_5$ is a second bull. So x sees z . Then x sees v , for otherwise $v - u_3xz - u_5$ is a second bull. Thus $v = u_2$, and $u \in N_2$. Then z sees every $u' \in N_2$, for otherwise $u' - u_3xz - u_5$ is a second bull. If there is any $y \in M_2$, then y sees u by (1), and z misses y , for otherwise $u_2 - uyz - u_5$ is a second bull; but then $y - u_3xz - u_5$ is a second bull. So $M_2 = \emptyset$ and z satisfies (a).

Now suppose that z sees u_3 and misses u_1 . Then z sees u_4 , for otherwise $u_1 - uz - u_3 - u_4$ is a second bull; and z sees u_5 , for otherwise $v - u_3zu_4 - u_5$ is a second bull; but then $u_1 - uu_3z - u_5$ is a second bull. Therefore, z misses u_3 . Thus z sees u_4 , for otherwise $z - uvu_3 - u_4$ is a second bull; and z misses u_1 , for otherwise z, u_1, v, u_3, u_4 induce a C_5 . If z sees x , then we must have $z \in A$, but then $u_1 - u_2xz - u_4$ is a second bull. So z misses x . Then x misses v , for otherwise $x - vu_1u - z$ is a second bull; and x sees u , for otherwise $x - u_3vu - z$ is a second bull. So $u = u_2$. Then z misses every $v' \in N_2$, for otherwise $x - u_2v'z - u_4$ is a second bull. If there is any $y \in M_2$, then y sees v by (1), and z sees y , for otherwise $z - u_2xu_3 - y$ is a second bull; but then $x - u_3vy - z$ is a second bull. Thus $M_2 = \emptyset$ and z satisfies (b). Therefore (2) holds.

By the hypothesis, there is a transitive orientation of $G - x$. In that orientation, we write $u \rightarrow v$ whenever the edge uv exists in $G - x$ and is oriented from u to v ; and for disjoint sets $Y, Z \subset V(G)$, we also write $Y \rightarrow Z$ if $y \rightarrow z$ holds for all $y \in Y$ and $z \in Z$. In the transitive orientation, we may assume up to symmetry that $u_i \rightarrow u_{i+1}$ for $i = 1, 3, 5$ and $u_i \rightarrow u_{i-1}$ for $i = 3, 5$. Then the transitivity implies $A \rightarrow u_2, u_3 \rightarrow B$, and $A \rightarrow B$. We claim that:

We may assume that every edge u_2v with $v \in U_2$ satisfies $v \rightarrow u_2$. (3)

To prove this, first suppose that $P(D) - x = \emptyset$. So U_2 is a homogeneous set in $G - x$. Moreover, by (1), every vertex of $\{u_2\} \cup M_2$ sees every vertex of N_2 . So we can reorient the edges between these two sets in such a way that $N_2 \rightarrow \{u_2\} \cup M_2$. Then it is easy to see that the modified orientation

is transitive. Now suppose that $P(D) - x \neq \emptyset$. So, by (2), we have $U_2 = \{u_2\} \cup N_2$. Let z be any vertex in $P(D) - x$. Suppose that z satisfies (a) of (2). Then the transitivity implies $\{u_1, u_3, u_5\} \rightarrow z$, and, consequently, $v \rightarrow z$ for every $v \in N_2$, and $v \rightarrow u_2$ as well. Thus we have the desired property. Finally, suppose that z satisfies (b). Then the transitivity implies $z \rightarrow \{u_2, u_4\}$ and consequently $v \rightarrow u_2$ for every $v \in N_2$. Thus we also have the desired property. Therefore (3) holds.

Let us extend this transitive orientation of $G - x$ to an orientation of G by setting $a \rightarrow x$ for every $a \in A \cup \{u_3\}$ and $x \rightarrow b$ for every $b \in B \cup \{u_2\}$. We claim that this is a transitive orientation of G . Note that there is no circuit in G , for if a set S of vertices induces a circuit, then S must contain x , and then (since $N(x) = \{u_2, u_3\} \cup A \cup B$ and $A \cup \{u_3\} \rightarrow B \cup \{u_2\}$) the set $S - x$ would induce a circuit in $G - x$. Now suppose that there is a triple r, s, t with $r \rightarrow s \rightarrow t$ and r, t are not adjacent. Clearly x is one of r, s, t , since the orientation is transitive in $G - x$. If $x = s$, then r is in $A \cup \{u_3\}$ and t is in $B \cup \{u_2\}$, but then we have $r \rightarrow t$ as mentioned above. So $x \neq s$. This leads to the following four cases.

Case 1: $x = t$ and $s \in A$. The transitivity (on r, s, u_2) implies $r \rightarrow u_2$. Suppose that r sees u_3 . The transitivity (on s, r, u_3) implies $r \rightarrow u_3$ and (on r, u_3, u_4) $r \rightarrow u_4$. Then r sees u_5 , for otherwise $x-u_3ru_4-u_5$ is a second bull; and r sees u_6 , for otherwise $u_2-ru_4u_5-u_6$ is a second bull; but then $x-u_3u_4r-u_6$ is a second bull. So r misses u_3 . Then r sees u_4 , for otherwise $r-u_2xu_3-u_4$ is a second bull. Then s sees u_4 , for otherwise r, s, x, u_3, u_4 induce a C_5 . Then r sees u_5 , for otherwise $x-sru_4-u_5$ is a second bull; and r sees u_6 , for otherwise $u_3-u_4ru_5-u_6$ is a second bull. But then $x-su_4r-u_6$ is a second bull.

Case 2: $x = t$ and $s = u_3$. The transitivity (on r, u_2, u_3, u_4) implies $r \rightarrow u_2$ and $r \rightarrow u_4$. Then r sees u_5 , for otherwise $x-u_3ru_4-u_5$ is a second bull; and r sees u_6 , for otherwise $u_2-ru_4u_5-u_6$ is a second bull. But then $x-u_3u_4r-u_6$ is a second bull.

Case 3: $x = r$ and $s \in B$. The transitivity (on u_3, s, t) implies $u_3 \rightarrow t$. If t misses u_2 , then it sees u_1 , for otherwise $u_1-u_2xu_3-t$ is a second bull; but then u_1, u_2, x, s, t induce a C_5 . So t sees u_2 . The transitivity (on s, t, u_2) implies $u_2 \rightarrow t$, and (on u_1, u_2, t) $u_1 \rightarrow t$. Then t misses u_4 , for otherwise $x-u_2u_1t-u_4$ is a second bull. But then $u_1-tsu_3-u_4$ is a second bull.

Case 4: $x = r$ and $s = u_2$. The transitivity (on u_1, u_2, t) implies $u_1 \rightarrow t$, and similarly we have $u_3 \rightarrow t$. then t misses u_j with $j \in \{4, 5\}$, for otherwise $x-u_2u_1t-u_j$ is a second bull; and t misses u_6 , for otherwise t, u_3, u_4, u_5, u_6 induce a C_5 . But now t is in U_2 , and the fact that $u_2 \rightarrow t$ contradicts (3). This completes the proof of the lemma. \square

Proof of Theorem 2

The proof of Theorem 2 goes by induction on the total number of sensitive vertices in G . We distinguish between two parts, (I) and (II).

(I) *First suppose that G has no sensitive vertex.* By Theorem 3, G admits a box partition. Consider any box B . If B contains any graph F_j with $j = 1, 2, 3$, then, using the auxiliary vertices a_B, b_B , we find a spiked F_j , which contradicts the fact that G is in \mathcal{B} . So B contains no bull and no lock. Gallai [10, 18] gave the list of all minimal forbidden subgraphs for the class of transitively orientable graphs. It is a routine matter to check that every forbidden subgraph in Gallai's list contains either an antihole, or a bull or a lock. It follows that every box B induces a subgraph that admits a transitive orientation $TO(B)$. Now we make an orientation of the edges of G by applying the rules below. In these rules we use the notation $u \rightarrow v$ to denote the orientation of an edge uv from u to v . Let us say that an edge uv in a box B is *sharp* if there is a vertex of $P(B)$ that sees exactly one of u, v , and *dull* otherwise.

- Rule 0: If uv is an edge where u is an odd vertex and v is an even vertex, then put $u \rightarrow v$.
- Rule S : If uv is a sharp edge in an odd box B , and there is a vertex of $P(B)$ that sees u and misses v , then put $u \rightarrow v$. In an even box, put $v \rightarrow u$.
- Rule $P3$: If uv is a dull edge in an odd box B , and there is a chordless path $u-v-w$ in B and a vertex of $P(B)$ that sees w and misses u, v , then put $u \rightarrow v$. In an even box, put $v \rightarrow u$.
- Rule $P4$: If uv is a dull edge in an odd box B , and there is a chordless path $u-v-w-z$ in B and a vertex of $P(B)$ that sees z and misses u, v, w , and vw is dull, then put $v \rightarrow u$. In an even box, put $u \rightarrow v$.
- Rule $Q3$: If uv is a dull edge in an odd box B , and there is a chordless path $u-v-q$ in B and a vertex of $P(B)$ that sees u, v and misses q , then put $v \rightarrow u$. In an even box, put $u \rightarrow v$.
- Rule $Q4$: If uv is a dull edge in an odd box B , and there is a chordless path $u-v-q-r$ in B and a vertex of $P(B)$ that sees u, v, q and misses r , then put $u \rightarrow v$. In an even box, put $v \rightarrow u$.
- Rule D : If a dull edge in a box B has not been oriented by the preceding rules, then orient it according to $TO(B)$.

Note that the rules give a symmetric role to odd boxes and even boxes. Let us prove that these rules produce a transitive orientation of G .

(10) *Every edge of G receives exactly one orientation.*

Clearly, Rules 0, S and D imply that every edge receives at least one orientation. Suppose that some edge uv receives the two opposite orientations $u \rightarrow v$ and $v \rightarrow u$. By Rule 0, edge uv is not between two boxes. If uv is a sharp edge, the opposite orientations must both be caused by Rule S , so there is a vertex of $P(B)$ that sees u and misses v and a vertex of $P(B)$ that sees v and misses u ; but this contradicts Lemma 4. So uv is a dull edge, say in an even box. It cannot be oriented in two opposite ways by Rule D , so each of the two opposite orientations is caused by Rules $P3, P4, Q3, Q4$. Up to symmetry this yields ten cases, which we analyse now. In either case we can consider the auxiliary vertices a_B, b_B for B . Suppose that the two opposite orientations are caused by:

- $P3$ and $P3$: So there is a chordless path $u-v-w-x$ with $w \in B$ and $x \in P(B)$, and there is a chordless path $v-u-z-y$ with $z \in B$ and $y \in P(B)$. If z misses w , then x misses z , for otherwise x, z, u, v, w induce a C_5 , and similarly y misses w ; but then $x-wva_B-z$ and $y-zua_B-w$ are two intersecting bulls. So z sees w . Then one of xz, yw is an edge, for otherwise zw is an edge that would be oriented in two opposite ways by Rule S , a contradiction. Say x sees z . Then b_B misses x , for otherwise $b_B-xzw-v$ and $b_B-xwz-u$ are two intersecting bulls. Then b_B-a_Bvw-x is a bull. Then b_B sees y , for otherwise b_B-a_Buz-y is a second bull. Then y misses w , for otherwise $b_B-ywz-u$ is a second bull. Then y sees x , for otherwise $y-zxw-v$ is a second bull. But then $b_B-yxz-u$ is a second bull, a contradiction.

- $P3$ and $P4$: So there is a chordless path $u-v-w-x$ with $w \in B, x \in P(B)$, and there is a chordless path $u-v-s-t-y$ with $s, t \in B, y \in P(B)$, and vs is dull. So x misses s . Then $u-a_Bst-y$ is a bull. Then w misses s , for otherwise $u-vsw-x$ is a second bull. But then vs is oriented in two opposite ways by Rules $P3$ and $P3$ (because of $x-w-v-s$ and $y-t-s-v$), a contradiction.

- $P3$ and $Q3$: So there is a path $u-v-w-x$ with $w \in B, x \in P(B)$ and a path $u-v-q$ with $q \in B$ and a vertex y that sees u, v and misses q . Note that either b_B-a_Bqv-y or $b_B-yuv-q$ is one bull. Then y sees w , for otherwise vw is a sharp edge oriented both ways by Rule S (because of x, y), which contradicts a fact already proved. Then x misses q , for otherwise vq is a sharp edge oriented both ways (because of x, y). Then x misses y , for otherwise $x-yuv-q$ is a second bull. Then w sees q , for otherwise $x-wyv-q$ is a second bull. But then $x-wqv-u$ is a second bull, a contradiction.

- $P3$ and $Q4$: So there is a chordless path $u-v-w-x$ with $w \in B, x \in P(B)$, and there is a chordless path $v-u-q-r$ with $q, r \in B$ and a vertex $y \in P(B)$ that sees v, u, q and misses r , and uq is dull. So x misses q . Note that either b_B-a_Brq-y or $b_B-yuq-r$ is one bull that contains q . Then x misses r , for otherwise rq is a sharp edge oriented both ways (because of x, y). Then y sees w , for otherwise vw is a sharp edge oriented both ways (because of x, y). Then w misses q , for otherwise uq is oriented both ways by $P3$ (because of $x-w-q-u$) and $Q3$ (because $u-q-r$ and y), which contradicts a fact already proved. Then x sees y , for otherwise $x-wvy-q$ is a second bull. But then

$x-yuq-r$ is a second bull, a contradiction.

- The remaining six cases ($P4$ and $P4$; $P4$ and $Q3$; $P4$ and $Q4$; $Q3$ and $Q3$; $Q3$ and $Q4$; $Q4$ and $Q4$) can all be treated as follows. When $u \rightarrow v$ is given by Rule $P4$, there is a chordless path $u-v-w-z-x$ with $w, z \in B, x \in P(B)$, and then $u-a_Bwz-x$ is a bull. When $u \rightarrow v$ is given by Rule $Q3$, there is a path $u-v-q$ in B and some $y \in P(B)$ that sees u, v and misses q ; then either b_B-a_Bqv-y or $b_B-yuv-q$ is a bull. When $u \rightarrow v$ is given by Rule $Q4$, there is a path $v-u-q-r$ in B and a vertex $y \in P(B)$ that sees v, u, q and misses r ; then either b_B-a_Brq-y or $b_B-yuq-r$ is a bull. And so when $v \rightarrow u$ is given by Rules $P4, Q3, Q4$, there is a similar bull. It is a routine matter to check that in each of the six cases, the two bulls produced by the two rules are distinct and intersect, a contradiction. Therefore (10) holds.

(11) *The orientation produced by the rules is transitive.*

Consider any chordless path $u-v-w$ in G . Assume that u, v, w are not all in the same box. Then, up to symmetry, one of them is odd and the other two are even (or vice-versa). If v is the odd one, we have $v \rightarrow u$ and $v \rightarrow w$ by Rule 0, so $u-v-w$ is oriented transitively. If u is the odd one, we have $u \rightarrow v$ by Rule 0 and $w \rightarrow v$ by Rule S , so $u-v-w$ is oriented transitively. Now we may assume up to symmetry that u, v, w are all in one odd box B . If both uv, vw are oriented by Rule D , then $u-v-w$ is oriented transitively since $TO(B)$ is a transitive orientation. So we may assume that at least one of uv, vw , say uv , is oriented by one of Rules $S, P3, P4, Q3, Q4$. Suppose by contradiction that the rules produce $u \rightarrow v$ and $v \rightarrow w$. In either case we can consider the auxiliary vertices a_B, b_B for B . Let us analyze all the cases.

- Suppose that the two orientations $u \rightarrow v$ and $v \rightarrow w$ are both caused by S . So there is a vertex $x \in P(B)$ that sees u and misses v , and there is a vertex $y \in P(B)$ that sees v and misses w . Then y sees u , for otherwise uv is oriented in two opposite ways by Rule S (because of x, y), which contradicts (10). Then x misses w , for otherwise vw is oriented both ways by S (because of x, y). Note that either $b_B-awv-y$ (if b_B misses y) or $b_B-yuv-w$ (if b_B sees y) is one bull. So x sees y , for otherwise $x-uyv-w$ is a second bull. But now, vertices u, v, w, x, y contradict Property (vii) for B .

So we may now assume, up to symmetry, that vw is dull.

- Suppose that $u \rightarrow v$ is caused by S . So there is a vertex $x \in P(B)$ that sees u and misses v . Then x misses w since vw is dull. But then $w \rightarrow v$ is given by Rule $P3$ (because of $x-u-v-w$), which contradicts (10).

- Suppose that $u \rightarrow v$ is caused by $P3$. So there is a chordless path $u-v-z-x$ with $z \in B, x \in P(B)$. Then x misses w since vw is dull. If z sees w , then $x-zwv-u$ and $x-zwa_B-u$ are two intersecting bulls, a contradiction. So z misses w . Then $w \rightarrow v$ is given by Rule $P3$ (because of $w-v-z-x$), which contradicts (10).

- Suppose that $u \rightarrow v$ is caused by $P4$. So there is a chordless path $v-u-z-p-x$

with $z, p \in B, x \in P(B)$, and zu is dull. So x misses w , since vw is dull. Note that $x-pza_B-v$ is one bull. If p sees w , then z sees w , for otherwise p, z, u, v, w induce a C_5 ; but then $x-pzw-v$ is a second bull. So p misses w . Then z sees w , for otherwise $x-pza_B-w$ is a second bull. But then $w \rightarrow v$ is given by Rule $P4$ (because of $x-p-z-w-v$), which contradicts (10).

- Suppose that $u \rightarrow v$ is caused by $Q3$. So there is a chordless path $v-u-q$ in B and a vertex $y \in P(B)$ that sees v, u and misses q . So y misses w , since vw is dull. If q sees w , then $w \rightarrow v$ is given by Rule $Q3$ (because of $v-w-q$ and y), a contradiction. If q misses w , then $w \rightarrow v$ is given by Rule $Q4$ (because of $w-v-u-q$ and y), a contradiction.

- Finally suppose that $u \rightarrow v$ is caused by $Q4$. So there is a chordless path $u-v-q-r$ in B and a vertex $y \in P(B)$ that sees u, v, q and misses r . Then y sees w since vw is dull. If w sees r , then $w \rightarrow v$ is given by Rule $Q3$ (because of $v-w-r$ and y), a contradiction. So w misses r . If w misses q , then $w \rightarrow v$ is given by Rule $Q4$ (because of $w-v-q-r$ and y), a contradiction. So w sees q . But then $r-qvw-u$ and $r-qwy-u$ are two intersecting bulls. Therefore (11) holds.

A classical theorem of Ghouila-Houri [11] states that if a graph admits a transitive orientation then it admits a transitive and acyclic orientation. So (11) suffices to prove our theorem and the proof of part (I) is complete. Actually, it is not hard to prove that the orientation produced by the above rules has no circuit, but we omit this proof.

(II) *Now G has a sensitive vertex.* Let x be any sensitive vertex of G . By the definition of a sensitive vertex, $G - x$ contains a hole of length at least six. Thus $G - x$ is in class \mathcal{B} and contains a hole. By the induction hypothesis, $G - x$ has a transitive orientation; and by Lemma 5, G has a transitive orientation. This completes the proof of Theorem 2. \square

4 A colouring algorithm

We conclude the paper with a discussion about how Theorem 1 indeed yields a polynomial-time algorithm that colours the vertices of any bull-reducible Berge graph containing no antihole.

We are given a bull-reducible Berge graph G with no antihole, with n vertices and m edges.

In the preliminary step, we will use the algorithm of Spinrad [20], which finds all maximal homogeneous sets of a graph. The complexity of Spinrad's algorithm is $O(mf(n, m))$, where $f(n, m)$ is the reverse of the Ackerman function. Remark that the maximal homogeneous sets are pairwise disjoint. For each such homogeneous set H , we can apply recursively our algorithm on H and find a coloring of H with $\omega(H)$ colors. Then we replace in G the vertices of H by a clique $Q(H)$ of size $\omega(H)$, and do this for each maximal

homogeneous set. Trivially the resulting graph is isomorphic to a subgraph of the original graph. At the end, it is easy to get a coloring of the original graph from a coloring of the new graph simply by merging the colors used on $Q(H)$ with the colors used in H .

In the second step, we determine whether the graph is weakly triangulated using the following “naive” method. For each triple abc forming a P_3 we test if this P_3 extends to a hole in the graph. Clearly, it suffices to check whether a and c are in the same component of the subgraph obtained from G by removing the vertices in $N(a) \cap N(c)$ and the vertices in $N(b) - \{a, c\}$. Using a shortest path algorithm we will find a shortest hole containing a, b, c , if any. Globally, we will either find that G is weakly triangulated or determine a shortest even hole in G .

If G is weakly triangulated, we refer to the algorithm in [15].

In the remaining case, if G has no sensitive vertex then G admits a box partition which can be determined by breadth-first search. Now, we may apply the rules to the box partition and obtain a transitive orientation for G . Then we apply the greedy method on the transitive orientation. Else, if G has a sensitive vertex x , then Lemma 5 extends a transitive orientation from $G - x$ to G .

The overall complexity is $O(n^4m)$.

The weighted case

Let us remark that this algorithm can be adapted to solve the weighted version of the coloring problem. Given a graph with a weight function w on its vertices, a *weighted coloring* is a family of stable sets S_1, \dots, S_q with weights $W(S_i)$ such that:

$$w(x) \leq \sum_{S_i \ni x} W(S_i) \quad (4)$$

holds for every vertex x . The goal is then to find a weighted coloring whose total weight $W(S_1) + \dots + W(S_q)$ is minimal. With our algorithm we can solve the minimum-weight coloring problem as follows.

If the graph has an incomplete homogeneous set H , we can recursively apply the algorithm on H and find a minimum-weight coloring for H . This consists of a family of stable sets T_1, \dots, T_p with weights $W(T_1), \dots, W(T_p)$. We then substitute H by a clique $Q(H)$ of cardinality p , resulting in a new graph G' . The i -th vertex in $Q(H)$ receives weight $W(T_i)$, while the vertices of $G' - Q$ keep the same weight as in the original graph. Classical polyhedral considerations (see [12]) imply that in a weighted perfect graph there exists a minimum-weight coloring that satisfies (4) with equality for every vertex. So, at the end, we can obtain a minimum-weight coloring of the original graph by combining a minimum-weight of the new graph with T_1, \dots, T_p .

When the graph is weakly triangulated, we can apply the minimum-weight coloring algorithm from [15].

When the graph is transitively orientable, we can apply the minimum-weight coloring algorithm from [16], whose complexity is $O(nm)$.

The overall complexity is again $O(n^4m)$.

To find a maximum-weight clique is straightforwardly similar, because the algorithms respectively from [15] and [16] can also be required to produce a maximum weighted clique in respectively a weakly triangulated graph and a transitively orientable graph.

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